Class 1, given on Jan 4, 2010, for Math 13, Winter 2010

1. Introductory Comments.

Instructor information. Name: Andrew Yang, Office: Kemeny 316, Email: Andrew.C.Yang@dartmouth.edu, Office phone: 646-2960, Office hours: Tuesday and Thursday, 1:00pm - 2:30pm, or by appointment (use email or talk to me in person).

Book: *Calculus*, 6th edition, by James Stewart. Available at the bookstore. This is the same book used in Math 8 in the fall. The website has some optional textbooks listed, along with relevant comments. In particular, the book *Div, grad, curl, and all that*, might be a useful supplement for the second half of the class.

Webpage: `www.math.dartmouth.edu/∼m13w10`. This is very important as it will have a link to Webwork as well as a your weekly written homework assignments. It has comprehensive information about the class and will also be the place to go for updates as the class proceeds.

Grading: Based on homework and exams. Homework will constitute 25% of the grade, while exams the remaining 75%. There will be two midterms, to be held at times to be determined in the near future. The final will be sometime in the middle of March, of three hours in length, with the exact date to be determined. Of the 75% of your grade determined by exams, the breakdown is 20/20/35.

There are two types of homework: Webwork and written assignments. Webwork is an entirely computer-based homework system, where you login (go to the math 13 webpage for a link to the login page) using your ID and password, and get homework problems from the Webwork system. You input your answers back into the computer, and the system will tell you if you are right or wrong. If you are wrong, you have the chance (infinitely many chances, most of the time) to find the correct answer. Each set of problems in the Webwork system will have a 'closing time', after which you will not be able to work on the problems for credit anymore. You will get a new Webwork assignment each class, so be sure to check the system daily, or at least every other day! Usually, Webwork assignments will be due at 10am four or five days after the corresponding content is covered in class.

You should receive a Webwork ID and password in the near future (within two days). Once you receive the automated email to your Dartmouth email account, test to make sure that you can login properly. If you are having difficulty logging in contact me immediately and I will try to solve your problem.

Written homework assignments are more traditional and will be due once a week, on Wednesday at 2pm. You can submit them to the homework boxes outside of the classroom or at the very beginning of class. Each week's homework will be listed on the course webpage. A grader will grade your assignments and they will usually be returned back in class, usually on the Monday after they have been turned in. On written assignments, we expect you to show all your work in a reasonably organized way. Correct answers without supporting justification will not be given full marks. Webwork will account for 10 of the 25 percentage points allocated to HW, while written homeworks will be the remaining 15 percent.

Tutorials: On Sunday, Tuesday, and Thursday nights, Zachary Hamaker will be running tutorial sessions in a time and location to be determined. We plan for the tutorial sessions to start on Sunday, Jan 10 2010, the day before the first homework assignment is due, on Monday, Jan 11 2010.

X-hour: The X-hour for this section is on Thursday, at 1:00pm - 1:50pm. Keep this slot of time available – although we will not use it regularly, there will be at least a few weeks where we will use the X-hour as a replacement class for another day where I will not be in, or when a holiday (Martin Luther King, Jr. and the Winter Carnival) occurs on one of the regularly scheduled class days.

Late homework policy: In general, unexcused late homeworks will not be accepted for credit, although you can turn in late homework assignments which can be returned with comments. The only general reasons we will grant extensions on homework are for illness or family emergencies. In these cases, please notify me before the assignment is due with the reason why you cannot turn in the assignment on time. If you have some other reason why you cannot finish an assignment on time, you can always email me and ask for an extension, although I cannot guarantee that you receive one. This late policy homework applies to both Webwork and written homeworks.

Assistance: In general, there is a good amount of assistance available for this class. There are the tutorial sessions, as well as office hours. If you are having trouble in the class, do not hesitate to seek help.

2. General advice

- The most important piece of advice is to keep up with the progress of the class. Mathematics may very well be the subject where progress at any one point is most dependent on understanding everything that came before it, so once you fall behind you will have difficulty understanding the material in subsequent classes.
- The best way to test yourself for understanding of mathematical content is to solve problems by yourself. Of course, if you are having some difficulty and are not making progress on a problem, you should feel free to seek assistance from classmates, TAs, or the instructors, but it is worth trying each problem on your own for at least some amount of time. Even if you do not find a solution you may benefit from partial progress on the problem, or discover your mathematical weaknesses.
- As a matter of fact, if you are really serious about learning mathematics, you should attempt to do every problem in the textbook. I don't mean this literally, but I do mean that you should try any problem which does not seem trivially easy to you - and in particular, this means all of the more difficult problems at the end of each section and chapter. Of course, this is by no means expected or required to get a good grade.
- The above being said, for the best understanding of mathematical material, you should not only work homework assignments, but also discuss mathematics (with peers, TAs, or instructors) verbally. You will find that speaking, listening, reading, and writing mathematics are all a bit different from each other, and that maximal understanding only comes about when you engage in all four of these activities.
- Browse the section of a textbook that will be covered in class prior to actually attending the class. You shouldn't expect complete understanding, but some exposure to the terms and ideas before attending class will probably make class more enlightening and less confusing.
- If you start to have difficulty, do not hesitate to seek help. As mentioned earlier, falling behind is highly undesirable, and there is plenty of available assistance.
- Try to do a little bit of mathematics everyday. Of course, you really don't have a lot of choice since there are homework assignments every other day, but doing mathematics isn't very different from playing a sport or a musical instrument – unless you have a lot of experience, you need to practice daily to stay at your best. We don't expect you to work on math two hours a day, but something more like twenty or thirty minutes per day, on average (outside of lectures, of course) isn't unreasonable.

3. An introduction to Math 13

Math 13 at Dartmouth is intended to be the sequel to Math 8. The second half of Math 8 covers differential calculus of functions of several variables, and so the entirety of Math 13 is devoted to studying integral calculus of functions of several variables. Today, we will briefly discuss the 'big-picture' of what you should expect to learn in Math 13, and how it is related to the calculus you learned in a first-year calculus class.

Recall that in single-variable calculus, there are either *definite* or *indefinite* integrals. The former is an expression which can be thought of as computing the area under the graph of a function, while the latter is simply an antiderivative; that is, a function whose derivative is equal to some other function. The relationship between these two concepts is given by the *Fundamental Theorem of Calculus*, which states that if $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This has probably become so ingrained in your mind that you no longer consciously think about the fact that, on the surface, definite and indefinite integrals aren't really similar at all. Indeed, a definite integral is defined as a limit of Riemann sums, which is a really unwieldy and abstract concept to work with.

If we want to generalize these ideas to functions of several variables, we will have our work cut out for us. Notice that we do not even have a notion of what it means for a function $F(x, y)$ to be the antiderivative of another function $f(x, y)$, or indeed, if this could even be made sense of. Notice that we have defined partial derivatives, and also the gradient of a function $f(x, y)$, but $\nabla f(x, y)$ is not a function which takes on real-values; rather, it is a vector-valued function on $\mathbb{R}^2$.

The first half of this class will be devoted to learning how to define and calculate definite integrals of functions of several variables. In analogy to the single-variable case, such definite integrals should represent volumes under the graphs of functions. However, one complication arises in these integrals which makes evaluating them substantially harder than definite integrals of a single variable. When we integrate a function $f(x)$, we do so over an interval $[a, b]$. However, we will want to integrate functions $f(x, y)$ over not just rectangles, but more general regions, such as circles, or other regions usually defined by functions we are familiar with. Learning how to deal with this additional complication will be the main new concept we will have to learn in the first half of the class.

The other parts of the first half of this class will be devoted to learning about how to use definite integrals in several different contexts. One such situation is when we want to integrate in coordinate systems more general than the familiar 'rectangular coordinates'. This has great practical applications when studying physics and engineering, since there are many instances where polar, cylindrical, or spherical coordinates are more 'natural' than rectangular coordinates. We will also look at how these integrals can be used to calculate quantities of interest in physics and engineering.

The second half of this class will be devoted to finding suitable analogues of the Fundamental Theorem of Calculus. To do so, we will have to learn about 'line' and 'surface' integrals, which arise naturally in physics when we want to calculate quantities such as work and flux. The class will culminate in the study of three great theorems, known as Green's Theorem, Stokes' Theorem, and the Divergence Theorem. These are all analogues of the Fundamental Theorem of Calculus in different situations, and while they all look quite different on the surface, closer inspection will reveal that they are related at a deep level.

A lot of the mathematics in this class is closely related to physics, and in particular, electromagnetism. Although we do not presuppose knowledge of any physics in this class, students taking a multivariable electromagnetism class will probably have an advantage since they will be basically doing twice as much of the same mathematics as students only taking a math class or a physics class. Some of our examples will be motivated by physics, but we will not be teaching physics, and will only use these examples to provide physical intuition for the mathematical concepts we will study.

4. Basic Mathematical Notation and Concepts

We conclude today's class with a quick primer on basic mathematical notation and concepts which you may or may not have seen before.

4.1. **Notation.** We will frequently write $\mathbb{R}$ for the set of real numbers, and $\mathbb{R}^n$ for the set of ordered n-tuples of real numbers. Geometrically, we identify $\mathbb{R}$ with a line, $\mathbb{R}^2$ with a plane, and $\mathbb{R}^3$ with three-dimensional space. We will often write $f : X \to Y$ to denote a function whose domain is X and which takes values in the set Y. For example, if we write $f : \mathbb{R} \to \mathbb{R}$, we are saying that f is a function which accepts real numbers and also outputs real numbers. This is in contrast to functions $f : \mathbb{R}^2 \to \mathbb{R}$ or $f : \mathbb{R} \to \mathbb{R}^2$, which are scalar valued functions of two variables, and vector-valued functions of one variable, respectively. Their graphs are a surface in $\mathbb{R}^3$ and a curve in $\mathbb{R}^2$, respectively.

We use curly brackets to delimit the elements of a set. For example, $\{0, 2, 4\}$ is the set of numbers $0, 2, 4$. We use the notation $x \in S$ to say that x is an element of the set S; eg, $0 \in \{0, 2, 4\}, 1 \notin \{0, 2, 4\}$. We may write $\{P(x)|x \in S\}$ to denote the set of all elements in S which satisfy an additional property $P(x)$; for example, $\{x > 0|x \in \mathbb{R}\}$ is the set of all real numbers which are greater than 0; ie, the set of all positive reals.

Given two sets X, Y, we write $X \cup Y$ for the *union* of X and Y, which is the set that consists of all elements either in X or Y. We write $X \cap Y$ for the *intersection* of X and Y, which is the set that consists of all elements in both X and Y. We write $X \subset Y$ to indicate that X is a *subset* of Y, which means that every element of X is an element of Y as well. We write $\emptyset$ to denote the empty set, which is the set that has no elements.

4.2. **Logical statements.** The language of mathematics is packed with statements of the form 'If P, then Q', where P, Q might be statements which can be either true or false. For example, the statement 'If S is a square, then S is a rectangle' is a true statement, while the statement 'If $f : \mathbb{R} \to \mathbb{R}$ is integrable, then f is continuous' is a false statement. Notice that to disprove this statement, we need only exhibit one counterexample; for instance, any piecewise continuous function (which itself is not continuous) will do. We write $\sim P$ for the *negation* of P. This is the statement which is false exactly when P is true, and can be obtained from P by adding the word 'not'. For example, if P is the statement 'S is a square', then $\sim P$ is the statement 'S is not a square'.

We bring all this up because the statement 'If P, then Q', can be expressed in several other ways. The mathematical shorthand for this statement is $P \implies Q$, which we sometimes read 'P implies Q'. Two other forms this statement might take are 'P is a sufficient condition for Q', or 'Q is a necessary condition for P'. The first statement says that as long as P is true, then Q is true, which is exactly what the original statement says, and the latter statement says that Q must be true if P is true: that is, if P is true, then Q is also true. Another way of saying 'If P, then Q' is by saying 'P is true only if Q is true', or more succinctly, 'P only if Q'. (The word 'only' is of critical importance, since omitting it completely changes the logical content of the statement.)

An important logical point is that the statement $P \implies Q$ **IS NOT ALWAYS EQUIVALENT** to the statement $Q \implies P$. We say two statements are equivalent if one is true exactly when the other is true. For example, consider the statement 'If S is a square, then S is a rectangle'. In this statement, P is 'S is a square', while Q is 'S is a rectangle'. Then it is obvious that $Q \implies P$ is false, since the statement 'If S is a rectangle, then S is a square' is clearly not true. The statement $Q \implies P$ is sometimes called the *converse* of $P \implies Q$.

On the other hand, $P \implies Q$ is logically equivalent to the statement $\sim Q \implies \sim P$, which is sometimes called the *contrapositive* of $P \implies Q$. This might not be obvious, but a careful consideration of what it means for $P \implies Q$ to be true, or consideration of a few examples, will explain why this statement is true.

Example. In the beginning of Math 8, we learned about the nth term test for divergence, which says that if $\{a_n\}$ is a sequence, then the series $\sum a_n$ diverges if $\lim a_n$ is not equal to 0. The converse of this statement is '$\sum a_n$ diverges implies $\lim a_n \neq 0$'. However, this is a false statement, as the sequence $a_n = 1/n$, which gives the harmonic series, shows. The series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges, but the sequence $\{a_n\}$ converges to the limit 0. This is why we kept emphasizing that the n term test can never be used to show that a series converges – only that it diverges.